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Supertraces on the Superalgebra of Observables of Rational Calogero Model based on the Root System

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Abstract

It is shown that the superalgebra $H_{W(R)}$ of observables of the rational Calogero model based on the root system R possesses Q_R supertraces, where Q_R is the number of conjugacy classes of the Coxeter group $W(R)$ generated by the root system R which have no eigenvalue -1.

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1 The superalgebra of observables.

The superalgebra $H_{W(R)}$ of observables of the rational Calogero model based on the root system R is defined in the following way.

For any nonzero $\vec{v} \in V = \mathbb{R}^N$ define the reflections $R_{\vec{v}}$ as follows:

$$R_{\vec{v}}(\vec{x}) = \vec{x} - 2\frac{(\vec{x},\, \vec{v})}{(\vec{v},\, \vec{v})}\vec{v} \qquad \text{for any } \vec{x} \in V. \tag{1}$$

Here $(\cdot, \cdot)$ stands for the inner product in V: $(\vec{x},\, \vec{y}) = \sum_{i=1}^{N} x_i y_i$, where the x_i are the coordinates of vector $\vec{x}$: $x_i \stackrel{def}{=} (\vec{x},\, \vec{e}_i)$, and the vectors $\vec{e}_i$ constitute an orthonormal basis in V: $(\vec{e}_i,\, \vec{e}_j) = \delta_{ij}$. The reflections (1) have the following properties

$$R_{\vec{v}}(\vec{v}) = -\vec{v}, \qquad R_{\vec{v}}^2 = 1, \qquad (R_{\vec{v}}(\vec{x}),\, \vec{u}) = (\vec{x},\, R_{\vec{v}}(\vec{u})), \qquad \text{for any } \vec{v},\, \vec{x},\, \vec{u} \in V. \tag{2}$$

The finite set of vectors $R \subset V$ is a *root system* if R is $R_{\vec{v}}$-invariant for any $\vec{v} \in R$ and the group $W(R)$ generated by all reflections $R_{\vec{v}}$ with $\vec{v} \in R$ (Coxeter group) is finite.

Let $\mathcal{H}^\alpha$ $(\alpha = 0, 1)$ be two copies of V with orthonormal bases a_i^α $(i = 1, \ldots, N)$, respectively. For every vector $\vec{v} = \sum_{i=1}^{N} v_i \vec{e}_i \in V$ let $v^\alpha \in \mathcal{H}^\alpha$ be the vectors $v^\alpha = \sum_{i=1}^{N} v_i a_i^\alpha$, so the bilinear forms on $\mathcal{H}^\alpha \otimes \mathcal{H}^\beta$ can be defined as

$$(x^\alpha,\, y^\beta) = (\vec{x},\, \vec{y}), \tag{3}$$

where $\vec{x},\, \vec{y} \in V$ and $x^\alpha,\, y^\alpha \in \mathcal{H}^\alpha$ are their copies. The reflections $R_{\vec{v}}$ act on $\mathcal{H}^\alpha$ as follows

$$R_{\vec{v}}(h^\alpha) = h^\alpha - 2\frac{(h^\alpha,\, v^\alpha)}{(\vec{v},\, \vec{v})}v^\alpha, \qquad \text{for any } h^\alpha \in \mathcal{H}^\alpha. \tag{4}$$

So the $W(R)$-action on the spaces $\mathcal{H}^\alpha$ is defined.

Let ν be a set of constants $\nu_{\vec{v}}$ with $\vec{v} \in R$ such that $\nu_{\vec{v}} = \nu_{\vec{w}}$ if $R_{\vec{v}}$ and $R_{\vec{w}}$ belong to one conjugacy class of $W(R)$. Consider the algebra $H_{W(R)}(\nu)$ of polynomials in the a_i^α with coefficients in the group algebra $\mathbb{C}[W(R)]$ subject to the relations

$$\begin{gathered} R_{\vec{v}} h^\alpha = R_{\vec{v}}(h^\alpha) R_{\vec{v}}, \qquad \text{for any } \vec{v} \in R, \qquad \text{and } h^\alpha \in \mathcal{H}^\alpha \\ [h_1^\alpha, h_2^\beta] = \varepsilon^{\alpha\beta}\left((\vec{h}_1,\, \vec{h}_2) + \sum_{\vec{v}\in R} \nu_{\vec{v}} \frac{(\vec{h}_1,\, \vec{v})(\vec{h}_2,\, \vec{v})}{(\vec{v},\, \vec{v})} R_{\vec{v}} \right) \text{ for any } h_1^\alpha,\, h_2^\alpha \in \mathcal{H}^\alpha. \end{gathered} \tag{5}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor, $\varepsilon^{01} = 1$.

This algebra is associative because it has faithful representation via Dunkl differential-difference operators [1] acting on the space of infinitely smooth functions on V. Namely, let

$$D_i = \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{\vec{v}\in R} \nu_{\vec{v}} \frac{v_i}{(\vec{x},\, \vec{v})}(1 - R_{\vec{v}}) \tag{6}$$

and [2, 3]

$$a_i^\alpha = \frac{1}{\sqrt{2}}(x_i + (-1)^\alpha D_i), \quad \alpha = 0, 1. \tag{7}$$

The reflections $R_{\vec{v}}$ transform the deformed creation and annihilation operators (7) as vectors:

$$R_{\vec{v}} a_i^\alpha = \sum_{j=1}^{N} \left(\delta_{ij} - 2\frac{v_i v_j}{(\vec{v},\, \vec{v})} \right) a_j^\alpha R_{\vec{v}}. \tag{8}$$

Since $[D_i,\, D_j] = 0$ [1], it follows that

$$[a_i^\alpha, a_j^\beta] = \varepsilon^{\alpha\beta} \left(\delta_{ij} + \sum_{\vec{v}\in R} \nu_{\vec{v}} \frac{v_i v_j}{(\vec{v},\, \vec{v})} R_{\vec{v}} \right), \tag{9}$$

which manifestly coincides with (5).

We call $H_{W(R)}(\nu)$ *the algebra of observables of Calogero model based on the root system* R.

The commutation relations (5) suggest to define the *parity* π by setting:

$$\pi(a_i^\alpha) = 1 \text{ for any } \alpha,\, i, \qquad \pi(g) = 0 \text{ for any } g \in W(R) \tag{10}$$

and consider $H_{W(R)}(\nu)$ as a superalgebra.

Obviously, $\mathbb{C}[W(R)]$ is a subalgebra of $H_{W(R)}(\nu)$.

Observe an important property of superalgebra $H_{W(R)}(\nu)$: the Lie superalgebra of its inner derivations [1] contains $\mathfrak{sl}_2$ generated by

$$T^{\alpha\beta} = \frac{1}{2} \sum_{i=1}^{N} \left\{ a_i^\alpha,\, a_i^\beta \right\} \tag{11}$$

which commute with $\mathbb{C}[W(R)]$, i.e., $[T^{\alpha\beta},\, R_{\vec{v}}] = 0$, and act on a_i^α as on $\mathfrak{sl}_2$-vectors:

$$\left[T^{\alpha\beta},\, a_i^\gamma \right] = \varepsilon^{\alpha\gamma} a_i^\beta + \varepsilon^{\beta\gamma} a_i^\alpha. \tag{12}$$

The restriction of operator T^{01} in the representation (7) on the subspace of $W(R)$-invariant functions on V is a second-order differential operator which is the well-known Hamiltonian of the rational Calogero model [4] based on the root system R [5]. The parameters $\nu_{\vec{v}}$ are the coupling constants of this model. One of the relations (11), namely, $[T^{01},\, a_i^\alpha] = -(-1)^\alpha a_i^\alpha$, allows one to find the wave functions of the equation $T^{01}\psi = \epsilon\psi$ via usual Fock procedure with the vacuum $|0\rangle$ such that $a_i^0|0\rangle$=0 for any i [3]. After $W(R)$-symmetrization these wave functions become the wave functions of Calogero Hamiltonian.

2 Supertraces on $H_{W(R)}(\nu)$.

Any linear complex-valued function $str(\cdot)$ on the superalgebra $\mathcal{A}$ such that

$$str(fg) \;\; = \;\; (-1)^{\pi(f)\pi(g)} str(gf) \tag{13}$$

for any $f, g \in \mathcal{A}$ with definite parity $\pi(f)$ and $\pi(g)$ is called a *supertrace*.

[1] Let $\mathcal{A}$ be arbitrary associative superalgebra. Then, the operators $\mathcal{D}_x$ which act on $\mathcal{A}$ via $\mathcal{D}_x(y) = [x,\, y\}$ (supercommutator) constitute the Lie superalgebra of *inner* derivations.

Every supertrace $str(\cdot)$ on $\mathcal{A}$ generates the invariant bilinear form on $\mathcal{A}$

$$B_{str}(f, g) = str(f \cdot g). \tag{14}$$

It is obvious that if such a bilinear form is degenerate, then the null-vectors (i.e., $v \in \mathcal{A}$ such that $B(v, x) = 0$ for any $x \in \mathcal{A}$) of this form constitute the two-sided ideal $\mathcal{I} \subset \mathcal{A}$.

The ideals of this sort are present in the superalgebras $H_{W(A_1)}(\nu)$ (corresponding to the two-particle Calogero model) at $\nu = k + \frac{1}{2}$ [6] and in the superalgebras $H_{W(A_2)}(\nu)$ (corresponding to three-particle Calogero model) at $\nu = k + \frac{1}{2}$ and $\nu = k \pm \frac{1}{3}$ [7] for every integer k. For all the other values of ν all supertraces on these superalgebras generate the nondegenerate bilinear forms (14).

It is easy to describe all supertraces on $\mathbb{C}[W(R)]$. Every supertrace on $\mathbb{C}[W(R)]$ is completely determined by its values on $W(R) \subset \mathbb{C}[W(R)]$ and the function *str* is a central function on $W(R)$, i.e., the function constant on the conjugacy classes.

Before formulating the theorem establishing the connection between the supertraces on $H_{W(R)}(\nu)$ and the supertraces on $\mathbb{C}[W(R)]$, let us introduce the grading E on the vector space of $\mathbb{C}[W(R)]$. Consider the subspaces

$$\mathcal{E}^\alpha(g) = \{h \in \mathcal{H}^\alpha : \quad gh = -hg\} \text{ for } g \in W(R). \tag{15}$$

Clearly, $\dim\ \mathcal{E}^0(g) = \dim\ \mathcal{E}^1(g)$. Set [2]

$$E(g) = \dim\ \mathcal{E}^\alpha(g). \tag{16}$$

Obviously, $E(g)$ is equal to the number of (-1) in the spectrum of matrix g.[3]

The following theorem was proved in [8] [4]:

Theorem 1. *Let $\mathcal{P}(g)$ be the projection $\mathbb{C}[W(R)] \to \mathbb{C}[W(R)]$ defined as*

$$\mathcal{P}(\sum_i \alpha_i g_i) = \sum_{i:\, g_i \neq \mathbf{1}} \alpha_i g_i \textit{ for } g_i \in W(R),\ \alpha_i \in \mathbb{C}. \tag{17}$$

Let the grading E defined in (16) and the subspaces $\mathcal{E}^\alpha(g)$ defined in (15) satisfy the equations

$$E(\mathcal{P}([h_0, h_1])g) = E(g) - 1 \quad \textit{for any } g \in W(R), \quad \textit{and } h_\alpha \in \mathcal{E}^\alpha(g). \tag{18}$$

Then every supertrace on the algebra $\mathbb{C}[W(R)]$ satisfying the equations

$$str([h_0, h_1]g) = 0 \qquad \textit{for any } g \in W(R) \textit{ with } E(g) \neq 0 \textit{ and } h_\alpha \in \mathcal{E}^\alpha(g), \tag{19}$$

can be uniquely extended to a supertrace on $H_{W(R)}(\nu)$.

It is shown below that conditions (18) hold for arbitrary Coxeter group $W(R)$ and the number of independent solutions of conditions (19) is equal to the number of conjugacy classes in $W(R)$ with $E(g) = 0$.

[2] It follows from Lemma 3 formulated below that $\rho(g) = E(g)|_{mod2}$ is a grading on the group algebra $\mathbb{C}[W(R)]$. It is well known parity of elements of the Coxeter group.

[3] Indeed, let $\vec{x} \in V$ be an eigenvector of orthogonal matrix $g \in W(R)$, i.e., $g\vec{x} = \lambda\vec{x}$. Then (5) implies the relation $gx^\alpha = \lambda^{-1}x^\alpha g$ in $H_{W(R)}(\nu)$.

[4] This theorem was proved for the case $R = A_N$ only but the proof does not depend on the particular properties of the symmetric group $S_N = W(A_{N-1})$.

3 Conditions (18) for an arbitrary Coxeter group.

Lemma 2. Let g be an orthogonal $N \times N$ matrix which has no eigenvalue -1, i.e., the matrix $g+1$ is invertible. Then the matrix $R_{\vec{v}}g$ has exactly one eigenvalue equal to -1.

To prove this lemma let us consider the equation $R_{\vec{v}}g\vec{x} + \vec{x} = 0$ or $g\vec{x} + R_{\vec{v}}\vec{x} = 0$ for eigenvector $\vec{x}$ corresponding to eigenvalue -1. Using the definition of $R_{\vec{v}}$ one has $g\vec{x}+\vec{x}-2\frac{(\vec{v},\vec{x})}{|\vec{v}|^2}v = 0$; hence, $\vec{x} = 2\frac{(\vec{v},\vec{x})}{|\vec{v}|^2}(g+1)^{-1}\vec{v}$. It remains to show that this equation has a nonzero solution. Let $\vec{v} = (g+1)\vec{w}$. Then $|\vec{v}|^2 = 2(|\vec{w}|^2 + (\vec{w},\, g\vec{w}))$ and $((g+1)^{-1}\vec{v},\, \vec{v}) = |\vec{w}|^2 + (\vec{w},\, g\vec{w})$. So the vector $\vec{x}_1 = 2\frac{1}{|\vec{v}|^2}(g+1)^{-1}\vec{v}$ is the only (up to a factor) solution.

Lemma 3. Let g be an orthogonal $N \times N$ matrix and $\vec{c}_i$ ($i = 1, \dots, E(g)$) be the complete orthonormal set of its eigenvectors corresponding to eigenvalue -1. Then
i) $E(R_{\vec{v}}g) = E(g) + 1$ *if $(\vec{v},\, \vec{c}_i) = 0$ for all i;*
ii) *if there exists an i such that $(\vec{v},\, \vec{c}_i) \neq 0$, then $E(R_{\vec{v}}g) = E(g) - 1$ and the space of the eigenvectors of $R_{\vec{v}}g$ corresponding to eigenvalue -1 is the subspace of span$\{\vec{c}_1, \dots, \vec{c}_{E(g)}\}$ orthogonal to $\vec{v}$.*

Let $\vec{c}_i$, $i = 1, \dots N$, be the complete orthonormal set of the eigenvectors of g, i.e. $g\vec{c}_i = \lambda_i\vec{c}_i$. Here $\lambda_i = -1$ for $1 \le i \le E(g)$. Let $x^i = (\vec{x},\, \vec{c}_i)$ for every vector $\vec{x}$. Consider the equation for the eigenvector $\vec{x} = \sum_1^N x^i\vec{c}_i$ corresponding to eigenvalue -1 of matrix $R_{\vec{v}}g$:

$$(\lambda_i + 1)x^i - 2\frac{(g\vec{x},\, \vec{v})}{|\vec{v}|^2}v^i = 0, \tag{20}$$

where $v^i = (\vec{v}, \vec{c}_i)$. It follows from (20) that either $v^i = 0$ for $1 \le i \le E(g)$ or $(g\vec{x},\, \vec{v}) = 0$. In the first case, one can consider the restriction of $R_{\vec{v}}$ and g onto the subspace spanned by $\vec{c}_i$ with $i > E(g)$ and apply Lemma 2 to this restriction and obtain *i)*. In the second case, it follows from equation (20) that $x^i = 0$ for $i > E(g)$, hence, $g\vec{x} = -x$, $(\vec{x},\, \vec{v}) = 0$ which yields *i)*.

Now one can prove the following

Theorem 4. *Let $g \in W(R)$. Let $c_1^\alpha, c_2^\alpha \in \mathcal{E}^\alpha(g) \subset H_{W(R)}$ (i.e. $gc_1^\alpha = -c_1^\alpha g$, $gc_2^\alpha = -c_2^\alpha g$). Let $\mathcal{P}(g)$ be the projection (17). Then*

$$E(\mathcal{P}([c_1^\alpha,\, c_2^\beta])g) = E(g) - 1 \quad \textit{for any } g \in W(R). \tag{21}$$

Proof easily follows from the formula

$$\mathcal{P}([c_1^\alpha,\, c_2^\beta]) = \varepsilon^{\alpha\beta}\sum_{\vec{v}\in R}\nu_{\vec{v}}\frac{(\vec{c}_1,\, \vec{v})(\vec{c}_2,\, \vec{v})}{(\vec{v},\, \vec{v})}R_{\vec{v}}\,. \tag{22}$$

Indeed, if $(\vec{c}_1,\, \vec{v})(\vec{c}_2,\, \vec{v}) \neq 0$, then Lemma 3 implies that $E(R_{\vec{v}}g) = E(g) - 1$.

4 The supertraces on $\mathbb{C}[W(R)]$, Ground Level Conditions and the number of supertraces on $H_{W(R)}(\nu)$.

Due to the $W(R)$-invariance, the definition of the supertrace on $\mathbb{C}[W(R)]$ is the definition of the central function on $\mathbb{C}[W(R)]$ i.e. a function constant on each conjugacy class of

$\mathbb{C}[W(R)]$. Thus the number of the supertraces on $\mathbb{C}[W(R)]$ is equal to the number of conjugacy classes in $\mathbb{C}[W(R)]$.

Since $\mathbb{C}[W(R)] \subset H_{W(R)}(\nu)$, some additional restrictions on these functions follow from (13) and the defining relations (5) for $H_{W(R)}(\nu)$. Indeed, consider some elements c_i such that $gc_i = -c_i g$, where $g \in W(R)$ and $c_i \in \mathcal{H}^0 \oplus \mathcal{H}^1$. Then, one finds from (13) and (15) that $str\,(c_i c_j g)= -str\,(c_j g c_i)= str\,(c_j c_i g)$ and, therefore, $str\,([c_i, c_j]g) = 0$.

Since $[c_i, c_j]g \in \mathbb{C}[W(R)]$, these conditions restrict supertraces of degree-0 polynomials in a_i^α. In [8] we called them Ground Level Conditions (GLC).

They express the supertrace of elements g with $E(g) = e$ via the supertraces of elements $R_{\vec{v}}g$ with $E(R_{\vec{v}}g) = e - 1$:

$$str(g) = -str(([c_i^0,\, c_i^1] - 1)g), \text{ if } (\vec{c}_i,\, \vec{c}_i) = 1. \tag{23}$$

Ground Level Conditions (19) is an overdetermined system of linear equations for the central functions on $\mathbb{C}[W(R)]$.

Let us prove by induction on $E(g)$ the following theorem

Theorem 5. *GLC (19) have nonzero solutions and the number of independent solutions is equal to the number of conjugacy classes in $W(R)$ with $E(g) = 0$.*

The first step is simple: if $E(g) = 0$, then $str(g)$ is an arbitrary central function. The next step is also simple: if $E(G) = 1$, then there exists a unique element $c_1^0 \in \mathcal{E}^0(g)$ and a unique element $c_1^1 \in \mathcal{E}^1(g)$ such that $|c_1^\alpha| = 1$ and $gc_1^\alpha = -c_1^\alpha g$. Since $(([c_1^0,\, c_1^1] - 1)g) \in \mathbb{C}[W(R)]$ and $E((([c_1^0,\, c_1^1] - 1)g) = 0$, then

$$str(g) = -str(([c_1^0,\, c_1^1] - 1)g) \tag{24}$$

is the unique possible value for $str(g)$ with $E(g) = 1$. A priori these values are not consistent with other GLC.

Suppose that Ground Level Conditions

$$str\left([c_i^0, c_j^1]g\right) = 0 \tag{25}$$

considered for all g with $E(g) \leq e$ and for all $c_i^\alpha \in \mathcal{E}^\alpha(g)$ $(i = 1, , \ldots, e)$ such that $(c_i^\alpha,\, c_j^\beta) = \delta_{ij}$ have Q_e independent solutions.

Statement 6 *The value Q_e does not depend on e.*

It was shown above that $Q_1 = Q_0$. Let $e \geq 1$. Let us consider $g \in W(R)$ with $E(g) = e + 1$. Let $c_i^\alpha \in \mathcal{E}^\alpha(g)$ $(i = 1, 2)$ be such that $(c_i^\alpha,\, c_j^\beta) = \delta_{ij}$. These elements give the conditions:

$$str(g) = -str(([c_1^0,\, c_1^1] - 1)g), \tag{26}$$
$$str(g) = -str(([c_2^0,\, c_2^1] - 1)g), \tag{27}$$
$$str([c_1^0,\, c_2^1]g) = 0. \tag{28}$$

Let us transform (26):

$$str(g) = str(S_1) - str(S_{12}), \text{ where} \tag{29}$$
$$S_1 = -\left([c_1^0,\, c_1^1] - 1 - \sum_{\vec{v}\in R:\, (\vec{v},\vec{c}_1)(\vec{v},\vec{c}_2)\neq 0} \nu_{\vec{v}} \frac{(\vec{v},\, \vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}}\right) g =$$

$$
\begin{aligned}
& - \left(\sum_{\vec{v}\in R:\ (\vec{v},\vec{c}_1)(\vec{v},\vec{c}_2)=0} \nu_{\vec{v}} \frac{(\vec{v},\,\vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g = \\
& - \left(\sum_{\vec{v}\in R:\ (\vec{v},\vec{c}_2)=0} \nu_{\vec{v}} \frac{(\vec{v},\,\vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g, \qquad (30) \\
S_{12} \;=\; & \left(\sum_{\vec{v}\in R:\, (\vec{v},\vec{c}_1)(\vec{v},\vec{c}_2)\neq 0} \nu_{\vec{v}} \frac{(\vec{v},\,\vec{c}_1)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g. \qquad (31)
\end{aligned}
$$

It is evident from (30) and Lemma 3 that $E(S_1) = e$ and $S_1 c_2^0 = -c_2^0 S_1$. Hence, due to (23) and inductive hypothesis

$$
str(S_1) = -str(([c_2^0,\, c_2^1]-1)S_1) = str(([c_2^0,\, c_2^1]-1)(([c_1^0,\, c_1^1]-1)g - S_{12})) \qquad (32)
$$

and as a result

$$
str(S_1) = str(([c_2^0,\, c_2^1]-1)([c_1^0,\, c_1^1]-1)g) - str(([c_2^0,\, c_2^1])S_{12}) + str(S_{12}). \qquad (33)
$$

Finally, (26) is equivalent under inductive hypothesis to

$$
str(g) = str(([c_2^0,\, c_2^1]-1)([c_1^0,\, c_1^1]-1)g) - str(([c_2^0,\, c_2^1])S_{12}). \qquad (34)
$$

Analogously, (27) is equivalent under inductive hypothesis to

$$
str(g) = str(([c_1^0,\, c_1^1]-1)([c_2^0,\, c_2^1]-1)g) - str(([c_1^0,\, c_1^1])S_{21}), \qquad (35)
$$

where

$$
S_{21} = \left(\sum_{\vec{v}\in R:\, (\vec{v},\vec{c}_1)(\vec{v},\vec{c}_2)\neq 0} \nu_{\vec{v}} \frac{(\vec{v},\,\vec{c}_2)^2}{|\vec{v}|^2} R_{\vec{v}} \right) g. \qquad (36)
$$

Now, let us compare the corresponding terms in (34) and (35). First, the relation

$$
str(([c_1^0,\, c_1^1]-1)([c_2^0,\, c_2^1]-1)g) = str(([c_2^0,\, c_2^1]-1)([c_1^0,\, c_1^1]-1)g) \qquad (37)
$$

is identically true for every (super)trace on $\mathbb{C}[W(R)]$, as $[c_1^0,\, c_1^1]$ commutes with g. Second,

$$
str(([c_1^0,\, c_1^1])S_{21}) = str(([c_2^0,\, c_2^1])S_{12}) \qquad (38)
$$

since

$$
str([c_1^0,\, c_1^1](\vec{v},\,\vec{c}_2)^2 R_{\vec{v}} g) = str([c_2^0,\, c_2^1](\vec{v},\,\vec{c}_1)^2 R_{\vec{v}} g) \qquad (39)
$$

for every $\vec{v} \in R$ such that $(\vec{v},\,\vec{c}_1)(\vec{v},\,\vec{c}_2) \neq 0$. Indeed, due to Lemma 3 the element

$$
\vec{c} = \beta_1 \vec{c}_1 + \beta_2 \vec{c}_2\,,\ \text{where } \beta_1 = -(\vec{v},\,\vec{c}_2) \neq 0 \text{ and } \beta_2 = (\vec{v},\,\vec{c}_1) \neq 0\,, \qquad (40)
$$

is orthogonal to $\vec{v}$:

$$
(\vec{v},\,\vec{c}) = 0 \qquad (41)
$$

and satisfies the relation

$$
R_{\vec{v}} g c^\alpha = -c^\alpha R_{\vec{v}} g \qquad (42)
$$

due to Lemma 3. This fact together with

$$E([c_i^0,\, c^1]R_{\vec{v}}g) = e - 1 \text{ for } i = 1, 2 \tag{43}$$

(this also follows from Lemma 3) and inductive hypothesis imply

$$str([c_i^0,\, c^1]R_{\vec{v}}g) = str([c^0,\, c_i^1]R_{\vec{v}}g) = 0 \quad (i = 1, 2). \tag{44}$$

Substituting $\vec{c}_1 = \frac{1}{\beta_1}(\vec{c} - \beta_2\vec{c}_2)$ and $\vec{c}_2 = \frac{1}{\beta_2}(\vec{c} - \beta_1\vec{c}_1)$ in the left-hand side of (39) and using (41) and (44) one obtains the right-hand side of (39). Thus, (26) is equivalent to (27); hence

$$str(([c_1^0,\, c_1^1] - 1)g) - str(([c_2^0,\, c_2^1] - 1)g) = 0 \tag{45}$$

for every orthonormal pair $c_1,\, c_2 \in \mathcal{E}(g)$. Consequently,

$$str([c_1^0,\, c_2^1]g) = 0 \tag{46}$$

which finishes the proof of Statement 6 and Theorem 5.

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